Student Strategies Suggesting Emergence of Mental Structures Supporting Logical and Abstract Thinking: Multiplicative Reasoning

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For many students, developing mathematical reasoning can prove to be challenging. Such difficulty may be explained by a deficit in the core understanding of many arithmetical concepts taught in early school years. Multiplicative reasoning is one such concept that produces an essential foundation upon which higher-level mathematical thinking skills are built. The purpose of this study is to recognize indicators of multiplicative reasoning among fourth-grade students. Through cross-case analysis, the researcher used a test instrument to observe patterns of multiplicative reasoning at varying levels in a sample of 14 math students from a low socioeconomic school. Results indicate that the participants fell into three categories: premultiplicative, emergent, and multiplier. Consequently, 12 new sublevels were developed that further describe the multiplicative thinking of these fourth graders within the categories mentioned. Rather than being provided the standard mathematical algorithms, students should be encouraged to personally develop their own unique explanations, formulas, and understanding of general number system mechanics. When instructors are aware of their students' distinctive methods of determining multiplicative reasoning strategies and multiplying schemes, they are more apt to provide the most appropriate learning environment for their students.

The difficulties encountered by students in their transition to advanced mathematical thinking may be explained by a lack of understanding of many concepts taught in early school years, especially multiplicative reasoning (Confrey, 1994; Dreyfus, 1991; Harel & Sowder, 2005). By its nature, advanced mathematical thinking relies on a cumulative foundation of prior mathematical experiences. Students cannot comprehend advanced mathematical topics such as differential equations unless they understand underlying concepts, such as differentiation. Differentiation requires conceptual understanding of the idea of functions and that assumes the student understands variables. Understanding variables is dependent upon the students' understanding of number, which is dependent upon understanding of quantification, which requires comprehension of serial correspondence. In other words, there is a structural order with each previous topic serving as a foundation for the next levels of mathematics, and therefore I focus here on investigating children's multiplicative reasoning.

Multiplicative Reasoning

The concept of unit with respect to addition is quite different than the concept of unit with respect to multiplication. Addition and multiplication involve hidden assumptions. The hidden assumption in addition is that the unit is one, which children readily understand, whereas the hidden assumption in multiplication is that the unit is one as well as more than one simultaneously (Chandler & Kamii, 2009). Splitting, Confrey's (1994) contribution to understanding multiplicative reasoning, speaks to conditions under which reunitizing occurs after the split in multiplication. Park and Nunes (2001) suggest that children's concept of multiplication originates in their schema of correspondences and not in the concept of addition. Under this definition, the concept of multiplication is defined by a constant relationship between two quantities known as ratio and is a core meaning of multiplicative reasoning. The ratio or rate is the constant unit that is called the multiplicand and acted upon by the multiplier. Children employ the schema of correspondence in order to represent fixed relationships between variables and solve multiplication problems.

Background Literature

Advanced mathematical thinking is viewed as beginning in elementary school with multiplicative reasoning (Clark & Kamii, 1996; Harel & Sowder, 2005; Piaget, 1965). Richardson, Berenson, and Staley (2009) in addition to Ball (2003) acknowledge the requirement for additional research concerning preservice as well as in-service teachers' understanding of multiplicative reasoning.

Cognitive Development of Multiplicative Reasoning

According to Clark and Kamii (1996), the cognitive development of multiplicative reasoning includes a key component: children learning to quantify. Referred to as qualitative quantification, children identify items as larger or smaller based on a given unit. For example, "Johnny got more of this pizza than me." The skill of qualitative quantification can occur with or without mastery of serial correspondence. If a student comprehends serial correspondence, he or she is able to order multiple items from largest to smallest based on the same unit. An example would be, "Johnny got more of this pizza than Sally who got more of this pizza than Jane." Qualitative quantification with serial correspondence generally appears first, often at or before kindergarten (Piaget, 1965).

Repeated Addition versus Schema of Correspondence

There are two different hypotheses suggested to explain the origin of multiplicative reasoning from the point of view of development and cognition. As a result, the use of varying models divided the research directions and caused a thinning rather than a concentration of research. The first hypothesis is that multiplicative reasoning is based upon repeated addition (Fishbein, Deri, Nello, & Marino, 1985; Steffe, 1994). The second hypothesis is that repeated addition is merely a procedure that can assist the student in achieving the correct answer, but understanding multiplicative reasoning lies in correctly perceiving the inherently multiplicative relationships between the objects being enumerated and the numbers representing those objects (Clark & Kamii, 1996; Nunes & Bryant, 1996; Piaget, 1965; Vergnaud, 1983, 1988).

Vergnaud (1988) envisioned multiplication as interconnected concepts and reached conclusions that are in agreement with those of Piaget and Steffe. Vergnaud concluded that multiplication does rely partly on addition but possesses an organizational structure that cannot be broken down entirely into additive structures.

Consider the problem, "How many cans of Coke will you have if you are given four six packs of Coke?" When presented with the problem $4 \times 6 = ?$ the student employing repeated addition would likely demonstrate that $6 + 6 + 6 + 6 = 24$ and arrive at the answer via counting. From a developmental perspective, counting to a solution is often a sign of additive strategies, not multiplicative strategies. When employing a multiplicative strategy, the student would "correspond" the six packs of Coke as units of 1 as well as units of 6 and would mentally or concretely note that there are 4 units of 6. The student may understand that the 4 is counting the units of 6. When the student is able to think simultaneously about units of one and units of more than one, the onset of multiplicative reasoning has occurred (Clark & Kamii, 1996).

Steffe (1994) contributed to the advancement of multiplicative reasoning research when he proposed that although the beginning of children's understanding of multiplication is in the construction of a unit that is repeatedly added, it is what goes on in the mind of the student before the repeated addition of the unit that is crucial for conceptual understanding of multiplicative reasoning. How the student understands the concept of unit and how the student relates the quantities within a problem is labeled a schema. Exactly how children develop their multiplicative reasoning (or schema) should be an area upon which teachers and researchers focus because children construct their multiplicative structures from their activities and persist in using their own ideas, despite the instruction of teachers (Olive, 2001; Steffe, 1994). It is important to understand the student's schema of numbers and multiplication as the depth of such understanding can lead to significant new ideas with respect to teaching mathematics to children (Confrey, 1991; Kieren, 1990; Kieren & Pirie, 1991; Piaget, 1973; Steffe & Cobb, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983).

The Path to Covariation

Originally, perhaps as far back as the early Greeks, the concept of number was based on counting. In those days, counting was considered sufficient for indicating the size or the magnitude (Smith & Confrey, 1994). Counting and measuring create a successor action known as addition. This concept of number is distinct from the concept of ratio. The early Greek philosopher Euclid described ratio as a comparison with respect to the size between two magnitudes (Behr, Lesh, Post, & Silver, 1983; Heath, 1956). The chief difference between these two concepts is that number is followed by a successor action of addition, whereas ratio is followed by a successor action of multiplication.

Rizzuti (1991) distinguishes between the traditional definition of function where the correspondence between two sets assigns to every element in the first set exactly one element in the second set, and the covariation that exists between two or more variables. The main idea of the difference between the two definitions of function is that the traditional definition makes the function more additive or discrete, that is, not continuous. In contrast, the idea that a function describes the covariation that exists between two or more variables makes the definition of function more multiplicative or continuous. Understanding functions as the correspondence between two (or more) variables requires students to learn a definition that is separated from the functional thinking they do outside of the mathematics class (Rizzuti, 1991; Smith & Confrey, 1994). Conceptualizing functions as covariation models of the relationship between two variables is foundational to student concept attainment surrounding logarithmic functions. For example, given the function $y = 2x$, the students who build two columns of a "T" table learn much from playing with the covarying actions discovered while populating such a table.

Smith and Confrey (1994) suggest that such "play" with "T" tables or covariation tables offers the potential of allowing students to construct their personal understanding of covariation and allows them to understand the covariation in the expressions they and others create. Kieren (1994) suggests that Smith's and Confrey's subjects are able to focus on actions and even actions on actions.

For example, we learn various rules such as $a^x a^y = a^{x+y}$ and $(a^x)^y = a^{xy}$, and then apply such rules to generate an answer via an algorithm. Unfortunately, the repeated application of such rules may lead many down a path of difficulty where we utilize the shortcut of repeated multiplication. This shortcut of repeated multiplication does not prepare us for the conceptual understanding needed to comprehend the next topics in mathematics, such as exponential and logarithmic functions. Confrey (1994) suggests that rather than depend on the actions of repeated multiplication, we should create independent multiplicative structures she calls splitting structures, arguing that because under the splitting model the function is understood with respect to covariation, the understanding of logarithmic functions will appear more naturally.

Smith (1994) argues that this propagation continues up the path to advanced mathematical thinking. He contends that Confrey's work indicates the need for further research involving rate of change, for example, the development of conceptual understanding of the mathematical constant *e*.

Summation

Often, students lack the mental structures necessary to fully comprehend multiplicative reasoning. Compounding the problem is the lack of exposure to multiplicative reasoning within the curriculum, as well as teachers' understanding of their students' reasoning. What is needed at this point is a framework within which to study levels of multiplicative reasoning.

Theoretical Framework

Beginning with the work of Clark and Kamii (1996), I identified five levels of reasoning. The five developmental levels described below emerged from a review of the literature and provided the framework for this study. The multiplicative reasoning levels, described by Thorton and Fuller, 1981; Karplus and Lawson, 1974; Clark and Kamii, 1996, are summarized in the paragraphs below, and also provide a framework for this study.

Researchers identify multiplicative reasoning as categorized in five levels beginning with spontaneous strategy, not yet additive/guessing. At the second level, additive strategy, the subject derives his/her answers utilizing addition or subtraction. At the third level, multiplicative strategy without success, the subject cannot make the transition from additive to multiplicative thinking, but understands that additive is not sufficient. Multiplicative strategy with success, the fourth level, uses multiplicative reasoning successfully with time to reflect in describing the relationship between the numbers. In the final level, proportional strategy, the subject introduces a new quantity as a unitizing factor and then successfully completes the problem through multiplication or division.

I propose an expansion of these five multiplicative reasoning markers to a more detailed list of markers. Toward this goal, I ask the following questions:

1. What are the indicators of multiplicative reasoning among fourth-grade students?
2. What strategies do fourth-grade students utilize in solving word problems that require multiplicative reasoning?

Method

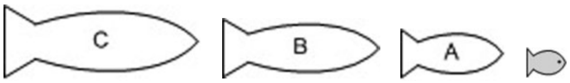
Participants

In cooperation with the administration of a large urban school district in the eastern part of the United States, the participants were recruited from the fourth grade at a Title I school. The participants were age 9 or 10 and were recommended for study by the school's math specialist as being among the strongest math students in fourth grade. Overall, 14 participants of varying ethnicity, nine of whom were female and five of whom were male, were engaged in the study.

Instrument

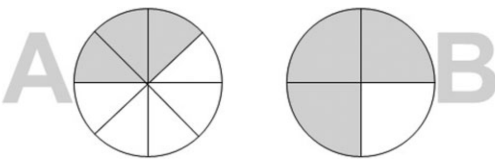
The participants were asked to engage in a dialog while being interviewed. Because the computer can provide an environment that can enhance children's own construction of multiplicative reasoning via interaction with the teacher (Olive, 2000), the computer was employed as a research tool. An instrument consisting of 10 questions was devised to invoke varying levels of multiplicative reasoning. Items were adapted from paper-and-pencil instruments developed by previous researchers (Clark & Kamii, 1996; Karplus & Lawson, 1974; Thorton & Fuller, 1981). The test instrument was developed in Microsoft Visio (Microsoft, Redmond, WA, USA) such that the participants were able to drag and drop smaller gray fish into larger fish to demonstrate their understanding of the multiplicative reasoning problems presented. Also provided were manipulatives that included a magnetic aquarium containing small green and large yellow fish, two pizza pans, and manipulatives such as paper clips, buttons, and stick figures for participants to demonstrate their understanding of test items. The test

Table 1
Multiplicative Reasoning Instrument

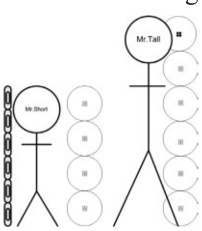


Refers to questions 1–8

Tank 1: Fish B is twice as large as Fish A. Fish A eats 2 gray fish each day. How many gray fish should you feed Fish B each day?
Tank 2: Fish A is fed 3 gray fish each day. Fish A is half the size of Fish B. How many fish should you feed Fish B each day?
Tank 3: Fish A is fed 2 gray fish each day. Fish B is 4 times larger than Fish A. How many gray fish do you feed Fish B each day?
Tank 4: Fish B is 3 times larger than Fish A. If Fish B eats 3 gray fish, how many gray fish should you feed Fish A each day?
Tank 5: Fish B is fed 9 gray fish each day. Fish A is one-third as big as Fish B. How many gray fish should you feed Fish A each day?
Tank 6: Fish C is 3 times larger than Fish A. Fish B is 2 times larger than Fish A. If Fish A is fed 2 gray fish each day, now many gray fish should you feed Fish B? How many gray fish should you feed Fish C?
Tank 7: Fish C is 4 times larger than Fish A. Fish C is 2 times larger than Fish B. If Fish C eats 8 gray fish each day, how many gray fish should you feed Fish A? Fish B?
Tank 8: Fish B is 2 times larger than Fish A. Fish C is 3 times larger than Fish B. If the Fish B eats 6 gray fish each day, how many gray fish will you feed Fish A? How many gray fish will you feed Fish C?



Tank 9. Assuming that both pizzas are equal in size, which pan has more pizza (the shaded portion)?



Tank 10. Mr. Short is 6 paper clips, or 4 buttons in height. Mr. Tall is 6 buttons in height. How many paper clips would it take to measure the height of Mr. Tall?

instrument included multiplicative reasoning problems similar to the following sample: “Tank 8: Fish B is 2 times larger than Fish A. Fish C is 3 times larger than Fish B. If Fish B eats 6 gray fish each day, how many gray fish will you feed Fish A? How many gray fish will you feed Fish C?” Some subject chose to use the hands-on approach utilizing the manipulatives provided while others were comfortable enough with the computer to drag and drop as a means to “feed” the larger fish. All were asked to explain their thought processes regarding their answers or strategies. The entire multiplicative reasoning instrument is contained in Table 1.

Data

The researcher employed two video cameras, one focused on the computer keyboard, computer screen, and hands of the participant, and the other focused on the participant’s written workspace. In addition to the video recordings, the participant responses were collected by the Microsoft Visio program. The participants also provided written work supporting their thinking during the interviews, and transcriptions of the participants’ verbal responses were made. By reviewing the video recordings, in conjunction with the Microsoft Visio files, the written work of the participants, and the transcriptions, the researchers explored indications of the emergence of participants’ multiplicative reasoning.

Data Analysis and Results

The data were examined by five coders for indications of the emergence of multiplicative reasoning utilizing the following techniques. Participants’ data were first coded with respect to the framework introduced in the theoretical framework section by observing the manner in which the participants responded to each question. Coded selections were then reexamined for key words, similarities, and differences within each of the five levels. When analyzing the

Table 2
Strategy Table for Level Mastery

Level	Strategy	Indicator	Reference
Level 1	Nonquantifier	Exhibits nonpreservation of the quantification of objects, i.e., A < B < C, meaning that 7 can = 8, can = 9	Clark and Kamii, 1996
Level 1	Spontaneous guesser	Arrives at answer through guessing	Clark and Kamii, 1996
Level 2	Key word finder	Derives answer from key words such as times and applies the associated algorithm	Sowder, 1988
Level 2	Counter	Enumerates objects with a one-on-one mapping with the whole number system	Dienes and Golden, 1966; Steffe, 1988
Level 2	Adder	Derives answer utilizing addition or subtraction regardless if strategy leads to success	Nunes and Bryant, 1996; Steffe, 1988; Thompson and Saldanha, 2003
Level 2	Quantifier	Makes use of the fact that A < B < C	Clark and Kamii, 1996; Lamon, 2006
Level 2	Measurer	Exhibits an understanding of when measurement should be linear or curvilinear and that each measurement has a starting and ending point without overlap or gap between unit measures	Kaput and West, 1994; Karplus, Pulos, and Stage 1983
Level 3	Repeated adder	Demonstrates an understanding that multiplicative answers can be achieved through repeated addition	Fishbein et al., 1985; Nunes and Bryant, 1996; Piaget, 1965; Steffe, 1994; Vergnaud, 1988
Level 3	Coordinator	Demonstrates limited ability to coordinate objects, numbers, and operations	Park and Nunes, 2001
Level 4	Multiplier	States a multiplication sentence and demonstrates fluency with respect to coordination of objects, numbers, and operations	Clark and Kamii, 1996
Level 4	Splitter	Utilizes concept of cutting and halving to indicate the need for division	Confrey, 1994
Level 5	Predictor	Predicts the measure of an object in one system given the measure of a proportional or similar object in another system	Kaput and West, 1994; Lamon, 2007

participants' data, it became apparent that there were indicators not found as descriptors in the original five levels listed in the theoretical framework section. These indicators were inserted into Table 2, and then the participants' data were reanalyzed using this expanded framework. In order for participants' indicators to be considered for entry into the expanded framework, the participants' indicators must have emerged from at least three of the five coders.

With the completion of analyses by the five coders, the researcher compiled the data into Table 2, described here. The researcher placed the participants' key words, transcripts, utterances, written work, schemes, and drawings for each question into one of the 12 strategies based on the following criteria found in the indicators column in Table 2. If the participant exhibited nonpreservation of the quantification of objects, then the participant was placed at level 1 nonquantifier. For example, if the participant placed 32 fish into a fish that could clearly not hold 32 fish, this would be an example of nonpreservation of the quantification of objects. If the participant arrived at the answer through guessing, then the participant was placed at level 1 spontaneous guesser. For example, a demonstration of level 1 spontaneous guesser behavior was exhibited when the participant noted that a particular fish should be fed seven fish because seven was his favorite number.

When the participant specifically indicated a need to multiply because the problem had the word "times" (or some other key word such as twice), the participant was placed at level 2 key word finder because the participant derived the answer by invoking the multiplication algorithm. When the participant counted the fish and uttered answers where it was easily observed that the answers were related to a one-on-one mapping with the whole number system, the participant was placed at level 2 counter.

Table 2 presents the strategies employed by the participants on the research instrument. The reference column in Table 2 identifies experts in the field who agree that the indicators in column three signify the emergence of multiplicative reasoning. Table 2 assists in answering research question one: What are the indicators of multiplicative reasoning among fourth-grade students?

Participants who arrived at their answers via addition, and plainly indicated so by writing or saying that they were

Table 3
Participant Strategies Frequency Table

	Level 5	Level 4A	Level 4B	Level 3A	Level 3B	Level 2A	Level 2B	Level 2C	Level 2D	Level 2E	Level 1A	Level 1B
1		2	6	1				1				
2			1		2	2		2	1		2	
3	1		5		1			1			2	
4			3				1	3	1	1	1	
5			3			1		2	1	2		1
6			3		2	2				3		
7			5		1	2				2		
8		1	3	1	1		1	2	1			
9			4	1		3	1	1				
10		1	5			2		2				
11	1	1	7			1						
12			2		1	2		4				1
13	1		5	1		1				2		
14		3	2	1		1				2		1
Σ	3	8	54	5	8	17	3	18	4	12	5	3

Note. In my research, I did not use A, B, C, D, or E behind the levels. They are introduced here only as a space saving technique. Level 5 = predictor; 4A = splitter; 4B = multiplier; 3A = coordinator; 3B = repeated adder; 2A = measurer; 2B = quantifier; 2C = adder; 2D = counter; 2E = key word finder; 1A = spontaneous guesser; 1B = nonquantifier.

adding more fish for each fish, were placed as a level 2 adder. When participants made good use of the fact that A < B < C, then the participants were placed as a level 2 quantifier. When participants could not find the answer via multiplication, they sometimes would measure the relative size of the fish, thus measuring the fish on the computer screen or the manipulatives and derive an answer through measurement. When the participants derived their answers via measurement, then the participants were placed as a level 2 measurer.

Sometimes the participants understood that additive strategies were not successful and attempted, but did not succeed, at utilizing multiplicative reasoning strategies. Often participants utilized repeated addition to obtain the correct answer. They would demonstrate this by writing or uttering 3 + 3 + 3 = 9, for example. When such repeated addition was utilized, the participants were placed as a level 3 repeated adder.

In many cases, the participant may have understood that additive strategies were not sufficient and succeeded at utilizing multiplicative reasoning by demonstrating a good ability to coordinate the objects, numbers, and operations defined within the word problem. Such participants often obtained the correct answer but did not or were not able to articulate the method or schema utilized to achieve the correct answer. Such demonstrations provided support for the participant being placed as a level 3 coordinator.

If the participant articulated an adequate mathematical sentence, either on paper or verbally, which fully described the mathematical relationship between the fish, the numbers, and the operations, then the participants would be placed as a level 4 multiplier. Additionally, some participants spoke or wrote the word "cut" or a similar word to indicate the need for division. In such a case, the participants were placed as a level 4 splitter.

When the participants provided a new quantity as a unitizing factor, articulated an adequate mathematical sentence, and successfully completed the problem using either multiplication or division, then the participants were placed as a level 5 predictor. In the case of question 10, the participants exhibiting level 5 predictor should have stated something similar to "6 is to 4 as y is to 6 and since 6/4 simplifies to 3/2 which equals 1.5 (this is the new unitizing factor) then $1.5 \times 6 = 9$."

The literature provides a clear baseline for understanding multiplicative reasoning outlined in the theoretical framework section. Yet the participant data reveal that a more detailed analysis can further identify the building blocks underlying the learning of mathematics for fourth-grade students. Table 2 identifies multiplicative indicators of these fourth graders along with additional strategies that expand the five levels of multiplicative reasoning to 12. These expanded strategies and indicators are then employed to reanalyze each of the 14 individual participant's performance on the 10-item instrument. The columns in Table 3 explain the frequency of the strategies used by the participants. The rows indicate the individual participant's use of strategies. This analysis addresses our second research question concerning strategies.

Examining the results reported in Table 3, I observed the frequency of individual participant's strategy use. It was interesting to note the spread of strategies and indicators occurring over these fourth-grade participants. Therefore, I sorted the participants' strategies into three categories. Those who were consistent in their choice of level 4 strategies or above (more than 70% of the time) I label as multiplicative reasoners. Premultiplicative are those participants who consistently chose strategies below level 4 in their attempts to solve the problems. Those participants who used a range of strategies I label as emergent and were also considered nonmultipliers.

Summary and Implications

The indicators of multiplicative reasoning are specific writings, utterances of key words, and behaviors of these participants as they engaged in problem solving. From these indicators, 12 strategies were identified that these fourth-grade participants utilized in solving multiplicative reasoning word problems. It was interesting to note that each of the 12 strategies was used at least three times. Separating the participants into three categories provided additional insights into the vast spread of multiplicative understandings present in low socioeconomic schools in fourth grade.

The practical implication of our findings is that they provide teachers with another tool to diagnose and assess students' understanding of multiplication. Teacher observations and informal assessment techniques can be used in conjunction with the list of indicators generated in Table 2. In effect, this list is another method for teachers to assist students' progress toward understanding multiplication and in determining the trajectories of students' multiplicative reasoning so that teachers can remediate and develop steps toward additional understanding (Ball, 2003; Richardson et al., 2009).

Conclusions and Discussion Section

Piaget's (1965) studies suggest that children form a network of mental structures to facilitate their understanding of the surrounding world and mathematics. Piaget listed stages of development of the typical child as: sensory motor stage (birth to age 2); preoperational stage (age 2–7); concrete operational stage (age 7–11); formal operations stage (age 11–16).

Throughout this study, I reviewed students' understanding of certain multiplicative reasoning word problems for fourth-grade students aged 9 and 10. Piaget states that around age 11 children are transitioning into the formal operations stage. One of the purposes of selecting fourth graders for the study is that some are just prior to entering and some are entering the formal operations stage.

Comprehending the beginnings of students' multiplicative reasoning gains significance when we understand that the roots of advanced mathematical thinking lie in the makeup of multiplicative reasoning (Tall, 1991). Given that forming relationships between objects and the associated values of those objects is the beginning of multiplicative reasoning (Lampert, 2001; Piaget, 1970; Vergnaud, 1988), and children's concepts of multiplication originate in their schema of correspondences and not in the concept of addition (Nunes & Bryant, 1996; Piaget, 1965; Vergnaud, 1983, 1988), it is important to understand each student's schema of numbers and multiplication because the depth of such understanding can lead to new ideas with respect to teaching mathematics to children (Confrey, 1991; Kieren, 1990; Kieren & Pirie, 1991; Piaget, 1973; Steffe & Cobb, 1988; Steffe et al., 1983). With advanced mathematical thinking being viewed as beginning in elementary school with multiplicative reasoning (Clark & Kamii, 1996; Harel & Sowder, 2005; Piaget, 1965; Tall, 1991), words that may be early indicators of multiplicative reasoning are: as (e.g., twice as big), area, split, half, one half, one third, divide, times, cut, more, less, double, larger, smaller, equal, sets, sets of sets, or their synonyms (Clark & Kamii, 1996; Confrey, 1994; Dienes & Golden, 1966; Inhelder & Piaget, 1958; Karplus, Pulos, & Stage, 1983; Steffe, 1994).

Conclusions

Table 4 provides the difference in proportion correct of the most and least proficient students (column four) and

Table 4
Difference in Most and Least Proficient Students

Item Number	Proportion Correct: Most Proficient Students	Proportion Correct: Least Proficient Students	Difference in Proportion Correct of Most and Least Proficient Students	Proportion Correct
1	1.00	1.00	0	.93
2	1.00	1.00	0	.93
3	1.00	1.00	0	1.00
4	1.00	.40	.60	.71
5	.80	0	.80	.43
6	1.00	.20	.80	.57
7	1.00	0	1.00	.64
8	1.00	.20	.80	.50
9	1.00	1.00	.00	1.00
10	.80	1.00	-.20	.93

proportion correct for each item on the test instrument (column five). A low value in column four indicates that the test item does not discriminate with respect to multiplicative reasoning. A high value in column five indicates that the item was very easy for these fourth-grade students.

From the findings of this study, I made the following conclusions. The multiplicative reasoning assessment instrument contained five questions (items 4, 5, 6, 7, and 8) that are appropriate discriminators among multiplicative reasoners, premultiplicative reasoners, and emergent multiplicative reasoners. These questions possess a discrimination index above .35, indicating that these items discriminate adequately between the high and low groups. Additionally, items 5, 6, 7, and 8 have excellent discrimination, possessing indices greater than .51.

The remaining test items, 1, 2, 3, 9, and 10, are poor discriminators of multiplicative reasoning. At first glance, the data suggest that these items are too easy for the subjects because the difficulty indices are well above .90; in reality, the reason for the high difficulty indices is that subjects used nonmultiplicative reasoning strategies that were successful. Thus, these items were not sufficient in leading subjects to utilize multiplicative reasoning strategies alone. For example, items 1–3 may invoke a multiplication strategy but not a multiplicative reasoning strategy simply because the key word "times" appears in the wording. In other words, it is quite possible to get the solutions to items 1–3 without using multiplicative reasoning, and the items do not discriminate between those who are using multiplicative reasoning and those who are using the multiplication algorithm.

Items 9 and 10 likewise have difficulty indices above .90, but only because obtaining the solution was arrived at by nonmultiplicative reasoning strategies. In other words, the subjects measured the manipulatives or the computer screen to obtain the solution. If the definition of solution, though, is modified to mean "expressed a succinct mathematical sentence fully describing the relationships among the numbers and operations present in the word problem," then the difficulty indices are greatly reduced, as indicated in Table 5. For item 9, the difficulty index decreases from 1.00 to .21, indicating that understanding item 9 is far more difficult than measuring it. Similarly for item 10, the index decreases from .93 to .07, indicating that understanding item 10 is far more difficult than obtaining the solution via measurement. Perhaps in future testing, the investigator should not allow measurement and counting as a choice of strategies when discriminating for multiplicative reasoning.

Table 5
Modified Difference in Most and Least Proficient

Item Number	Proportion Correct: Most Proficient Students	Proportion Correct: Least Proficient Students	Difference in Proportion Correct of Most and Least Proficient Students	Proportion Correct
1	1.00	1.00	0	.93
2	1.00	1.00	0	.93
3	1.00	1.00	0	1.00
4	1.00	.40	.60	.71
5	.80	0	.80	.43
6	1.00	.20	.80	.57
7	1.00	0	1.00	.64
8	1.00	.20	.80	.50
9	.20	.20	0	.21
10	0	0	0	.07

The findings of this study suggest that mathematics teachers should not only look at whether answers are right or wrong, but also consider the understanding behind the solutions children derive. By expanding the marker table for level mastery (Clark & Kamii, 1996) from its original five to its current 12 levels and strategies, this researcher demonstrated that students can obtain the correct answer by several incorrect strategies, as well as correct ones. Therefore, it becomes the responsibility of the listening, caring teacher to guide students utilizing incorrect strategies (yet achieving the correct answer) to a path using a correct strategy.

Future Research

This research suggests that the children in this study gravitated toward one or more of the 12 strategies of mathematical thinking. The levels were created as a result of the utterances, behaviors, and writings of these students. More research is needed to identify additional areas surrounding the development of multiplicative reasoning, which can affect a student's progress toward advanced mathematical thinking. What are the critical ages with respect to mathematical development and the development of multiplicative reasoning? What are the clues and signs that a child or a group of children are at a particular developmental stage? How can we construct a reliable test to discriminate multiplicative reasoning from those subjects who are doing multiplication by a memorized algorithm? In what manner should such a test instrument be administered to allow for identification of students' misconceptions? How can identification of multiplicative reasoning levels suggest the necessary curriculum that promotes the development of mathematical reasoning? How can a test instrument be constructed in order to identify the indicators of multiplicative reasoning among fourth-grade students? The answers to these questions have potential to provide important markers in the development of students' multiplicative reasoning and a narrowing of the achievement gap in mathematics.

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